Induced Forces in Annular Magneto-Fluid Dynamic Traveling Wave Devices

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Formulated and analyzed here is the problem of electromagnetic propagation through a circular cylindrical traveling-wave tube, infinite in extent in both directions, and inside of which a conducting fluid of arbitrary, but constant, conductivity is moving axially in an annulus with uniform speed U Asymptotic solutions valid for $kR_1 \ll 1$ and $kR_1 \gg 1$, where k is the wave number and R_1 the inner radius, are obtained and analyzed Concentrating specifically on the axial body force induced in the fluid, the $kR_1 \gg 1$ solution indicates, fixing the current sheet input and geometry, that the magnitude of the induced axial force is a marked function of five dimensionless parameters The most important of these is the parameter β, which represents the fluid magnetic Reynolds number based on wave speed and wavelength multiplied by the slip Fixing the remaining parameters, the body force rises linearly with β for small β, reaches a peak, and then, because of the inability of the electromagnetic field to penetrate into the conductor with increasing β (skin effect), falls exponentially for large β From the practical viewpoint, the results suggest the adjustment of the electrical and fluid mechanical parameters so that operation occurs at those values which optimize the force It is further indicated that, for efficient utilization and conservation of working fluid mass and electrical eneıgy, injection and acceleration take place in an annulus of the tube close to the coils with the coıe region filled with laminated magnetic material of high permeability ratheı than inject and accelerate throughout the entire cross section

Nomenclature

A	=	azimuthal component of vector potential
$\mathbf{B}$	=	magnetic induction
$\mathbf{H}$	=	magnetic intensity
$\mathbf{E}$	=	electric intensity
$\mathbf{j}$	=	current density
$\mathbf{F}$	=	force per unit volume
G	=	dimensionless axial force per unit volume evaluated at inner radius
R_0,R_1	=	outer and inner radii, respectively, of annular traveling wave tube
U	=	axial speed of conducting fluid
V	=	wave speed
NI	=	number of ampere turns per unit length
k	=	wave number
ω	=	circular frequency
σ	=	electrical conductivity
μ	=	permeability
μ_0	=	free-space permeability
ϕ	=	scalar potential
i	=	$(-1)^{1/2}$
δ	=	ratio of free space to core permeability
α	=	$\alpha_1 + i\alpha_2$ = complex separation constant in region 2
$R_{m\lambda}$	=	fluid magnetic Reynolds number based on wave speed and wavelength
S	=	$(V - U)/V$ = slip
β	=	$R_{m\lambda}S$
β_1,β_2	=	functions of β
γ	=	$\gamma_1 + i\gamma_2$ = complex separation constant in region 3
ϵ	=	core magnetic Reynolds number based on wave speed and wavelength
ϵ_1,ϵ_2	=	functions of ϵ
ρ	=	Euler s constant
I_0,I_1	=	modified Bessel functions of the first kind of order zero and one, respectively
K_0,K_1	=	modified Bessel functions of the second kind of order zero and one respectively
∇	=	gradient operator
r,θ,z	=	cylindrical coordinates
$\hat{\mathbf{r}},\hat{\boldsymbol{\theta}},\hat{\mathbf{z}}$	=	unit vectors corresponding to the cylindrical coordinate directions
t	=	time
R_e	=	real part of
1,2,3	=	subscripts designating respective regions

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I Introduction

TRAVELING wave mechanisms, which inductively transfer electromechanical energy to or from a moving conducting fluid, are suitable for an increasing number of important and interesting applications, such as low thrust high specific impulse plasma accelerators,[1–5] a c generators,[6,7] pumps,[8 9] plasma heaters,[10] etc The attractive feature of these devices is that they are electrodeless eliminating all the inherent losses and problems when electrodes are wetted by the conducting fluid

The operation of a typical traveling wave device may be described simply as follows: Consider a conducting fluid of arbitrary but constant conductivity moving axially with uniform speed U inside a circular cylindrical tube (or in an annulus bounded by two concentric cylinders, see Fig 1) infinite in extent in both directions and of radius R_0 Impressed on a transmission line, in the form of a coil wound around the cylinder, is a purely sinusoidal traveling current sheet of the form $NI \exp i(kz - \omega t)\hat{\boldsymbol{\theta}}$ where NI is the amplitude of the number of ampere turns per unit length, k the wave number, ω the circular frequency, and $\omega/k = V$ the wave speed If the relative speed between the moving conductor and wave is different from zero, than from the circular symmetry, closed azimuthal currents will be induced in the conductor These currents, when crossed with the radial and axial components of the traveling $\mathbf{B}$ field, associated with the traveling current sheet produce axial and radial body forces, respectively As will become clear from the text, when $V > U$, the axial body force averaged over a period is in the flow direction, and the device is used as an accelerator or pump If $V < U$, the average axial body force is decelerating, and the device, when externally loaded, is used as a generator of alternating voltage Used as an accelerator, the total integrated axial body force over the interior of the channel is a measure of the thrust produced by the device

From the simple description of the interaction just described and from the specific nature of the propagation of an electromagnetic field inside a conductor, one is led to the following three conjectures concerning the magnitude and nature of the induced axial body forces:

1) It would appear that for the given current input $NI \exp i(kz - \omega t)$ (actually this current form is a super-position of the traveling current produced by the primary source and that mutually induced back on the coils), all that is required for the production of large body forces anywhere inside the channel is to increase the magnitude of the currents induced by operating with fluids of large conductivity However, whereas the induced currents may be large in the region close to the coils, the induced currents would attenuate rapidly toward the axis because of the skin effect (i e , the inability of the electromagnetic field to penetrate substantially into a good conductor) Besides the conductivity dependence, the induced forces at a given radial position may also depend strongly on the other operating parameters, i e , ω, k, and U Since large conductivity is at once aiding and hindering the induction of large body forces over the cross section, it appears, given the geometry and current sheet input, that the operating parameters both for the transmission line (k,ω) and the fluid conductor (σ, U) (where U, for example, may represent the injection speed into the device) may be judiciously chosen so as to produce the largest over-all electromechanical coupling

2) From the point of view of application to space propulsion, where efficient utilization and conservation of propellant mass and electrical energy is imperative, the following suggests itself: From the symmetry of the magnetic field distribution, the radial component of the magnetic field is zero on the axis independent of the skin effect so that the induced axial forces are zero or very small in the centerline region This, coupled with the natural attenuation associated with the skin effect, suggests that the most efficient way of operating a traveling wave device is to inject and accelerate the working fluid within an annulus bounded from above by the coil walls and below by a solid coaxial cylindrical wall and not to inject and accelerate over the entire cross section of the tube

3) Having once decided to operate in an annulus, advantage can be taken of the existence of a core region to substantially increase the magnitude of the induced axial forces This can be done by inserting into the hollow core a material of high magnetic permeability, appropriately laminated to prevent the flow of eddy currents Since the induced axial force depends on the magnitude of the radial component of the magnetic field, decreasing the reluctance in this manner should increase the flux of the radial field component throughout the region of the working fluid

With no restrictions whatever placed on the magnitudes of the operating parameters, the aim of this paper is to investigate analytically the electromagnetic propagation inside a traveling wave device whose geometry is schematically given by Fig 1 with particular emphasis on the nature and magnitude of the induced axial body force The formulation will be general and flexible enough so that the conjectures as outlined above may be examined and analyzed in great detail Since the fluid is assumed to move with constant velocity through the channel, our only assumption (besides the usually assumed ones of electrical neutrality and constant fluid properties) is the uncoupling of the fluid mechanical equations from the electromagnetic equations However, since our primary aim is to estimate the extent to which the skin phenomenon effects the induced forces, the variable motion of the conductor through the channel should not seriously alter the over-all conclusions to be drawn

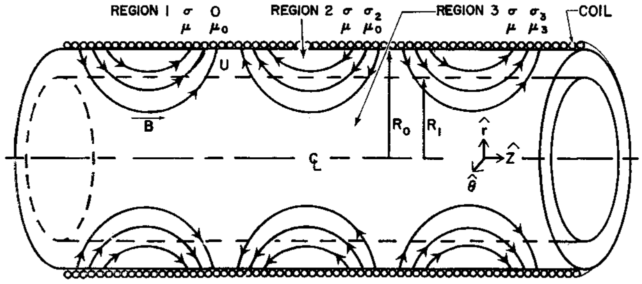

Fig 1 Schematic diagram of annular circular cylindrical traveling wave tube

II Formulation of the Problem

A Derivation of the Partial Differential Equations for the Vector Potentials

Let us refer to the schematic diagram of the circular cylindrical traveling wave tube shown in Fig 1 The nonconducting region with free-space permeability μ_0 outside of and surrounding the tube is designated as region 1 The annular region between the two coaxial cylinders R_0 and R_1 ($R_0 > R_1$) in which the working fluid of conductivity $\sigma = \sigma_2$ and permeability $\mu_2 = \mu_0$ is moving with uniform axial speed U is designated as region 2 The core region of conductivity $\sigma = \sigma_3$ and permeability $\mu = \mu_3$ is designated as region 3 Considering region 2 first, Maxwell's equations in mks units, neglecting the displacement current, are

$$\nabla \times \mathbf{E}_2 = -(\partial \mathbf{B}_2/\partial t) \tag{1}$$

$$\nabla \times \mathbf{H}_2 = \mathbf{j}_2 \tag{2}$$

and

$$\nabla \cdot \mathbf{B}_2 = 0 \tag{3}$$

Ohm's law in region 2 is

$$\mathbf{j}_2 = \sigma(\mathbf{E}_2 + U\hat{\mathbf{z}} \times \mathbf{B}_2) \tag{4}$$

where $U\hat{\mathbf{z}}$ is the constant axial velocity of the moving conducting fluid From Eq (3) and axial symmetry, a vector potential can be introduced having only an azimuthal component, i e ,

$$\mathbf{B}_2 = \nabla \times (A_2\hat{\boldsymbol{\theta}}) \tag{5}$$

so that the components of $\mathbf{B}_2$, in cylindrical coordinates, are

$$B_{2r} = -\frac{\partial A_2}{\partial z} \qquad B_{2z} = \frac{1}{r}\frac{\partial}{\partial r}(rA_2) \qquad B_{2\theta} = 0 \tag{6}$$

Substituting Eq (5) into Eq (1), and solving for $\mathbf{E}_2$, we obtain, in general,

$$\mathbf{E}_2 = -(\partial A_2/\partial t)\hat{\boldsymbol{\theta}} - \nabla\phi_2 \tag{7}$$

where ϕ_2 is some scalar potential Again, from the symmetry of the problem and the condition that there are no primary sources inside region 2, $\nabla\phi_2 = 0$ $\mathbf{E}_2$ is then purely azimuthal and is given by

$$\mathbf{E}_2 = -(\partial A_2/\partial t)\hat{\boldsymbol{\theta}} \tag{8}$$

The current density $\mathbf{j}_2$ is also purely azimuthal and, in terms of the vector potential, is given, from Eq (4), by

$$\mathbf{j}_2 = -\sigma_2\left(\frac{\partial A_2}{\partial t} + U\frac{\partial A_2}{\partial z}\right)\hat{\boldsymbol{\theta}} \tag{9}$$

Using the constitutive relation, $\mathbf{B}_2 = \mu_0\mathbf{H}_2$, and substituting Eq (5) and Eq (9) into Eq (2), we obtain the following partial differential equation for A_2:

$$\frac{\partial^2 A_2}{\partial r^2} + \frac{1}{r}\frac{\partial A_2}{\partial r} - \frac{A_2}{r^2} + \frac{\partial^2 A_2}{\partial z^2} = \mu_0\sigma_2\left(\frac{\partial A_2}{\partial t} + U\frac{\partial A_2}{\partial z}\right) \tag{10}$$

Similarly, the corresponding partial differential equation for A_1, in the nonconducting region 1, is

$$\frac{\partial^2 A_1}{\partial r^2} + \frac{1}{r}\frac{\partial A_1}{\partial r} - \frac{A_1}{r^2} + \frac{\partial^2 A_1}{\partial z^2} = 0 \tag{11}$$

Finally, in the core where the flow velocity is zero, the corresponding partial differential equation for A_3 is

$$\frac{\partial^2 A_3}{\partial r^2} + \frac{1}{r}\frac{\partial A_3}{\partial r} - \frac{A_3}{r^2} + \frac{\partial^2 A_3}{\partial z^2} = \mu_3\sigma_3\frac{\partial A_3}{\partial t} \tag{12}$$

B Boundary Conditions

If we assume a traveling current sheet of the form $NI \exp i(kz - \omega t)\hat{\boldsymbol{\theta}}$ acting in the coil where NI is the number of ampere turns per unit length, the boundary conditions at the coil $r = R_0$ become $\hat{\mathbf{r}}\ [\mathbf{B}] = 0$ and $\hat{\mathbf{r}} \times [\mathbf{H}] = NI \exp i(kz - \omega t)\hat{\boldsymbol{\theta}}$ where $\hat{\mathbf{r}}$ is the unit normal to the surface $r = R_0$ and [ ] represents the jump of the quantity in the brackets Similarly, since there is no circulating surface current sheet at the interface between regions 2 and 3, the boundary conditions at $r = R_1$ become $\hat{\mathbf{r}}\ [\mathbf{B}] = 0$ and $\hat{\mathbf{r}} \times [\mathbf{H}] = 0$

In terms of the vector potentials, the boundary conditions become at $r = R_0$,

$$\frac{\partial A_1}{\partial z} = \frac{\partial A_2}{\partial z} \tag{13}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rA_2) - \frac{1}{r}\frac{\partial}{\partial r}(rA_1) = \mu_0 NI \exp i(kz - \omega t) \tag{14}$$

and at $r = R_1$,

$$\frac{\partial A_2}{\partial z} = \frac{\partial A_3}{\partial z} \tag{15}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rA_2) - \delta\frac{1}{r}\frac{\partial}{\partial r}(rA_3) = 0 \tag{16}$$

where we have made use of the constitutive relations and $\delta = \mu_0/\mu_3$, i e , the ratio of the free-space permeability to the core permeability It is also required that

$$A_3 \text{ be finite at } r = 0 \tag{17}$$

and

$$A_1 \to 0 \text{ as } r \to \infty \tag{18}$$

III Exact and Asymptotic Solutions for the Complex Vector Potential A_2

A Exact Solution

In what follows, we shall use the same symbol A for the r dependent part of the vector potential This should not cause any confusion Seeking a solution in region 2 of the form $A_2(r) \exp i(kz - \omega t)$ and substituting into Eq (10), we obtain

$$\frac{d^2 A_2}{dr^2} + \frac{1}{r}\frac{dA_2}{dr} - \left(\frac{1}{r^2} + \alpha^2\right)A_2 = 0 \tag{19}$$

where the separation constant α, using $\omega/k = V$, is given by

$$\alpha = k\left\{1 - i\left(\frac{\mu_0\sigma_2\omega}{k^2}\right)\left(\frac{V - U}{V}\right)\right\}^{1/2} = k(1 - i\beta)^{1/2} \tag{20}$$

In Eq (20), the first quantity in parentheses, i e , $\mu_0\sigma_2\omega/k^2$, represents the fluid magnetic Reynolds number $R_{m\lambda}$ based on wave speed and wavelength The second quantity in parenthesis $(V - U)/V$, represents the slip S We have defined the product $R_{m\lambda}S = \beta$ Defining the complex separation constant α by $\alpha = \alpha_1 + i\alpha_2$, then from Eq (20) we obtain for the real and imaginary parts

$$\alpha_1 = k\beta_1 \qquad \alpha_2 = k\beta_2 \tag{21}$$

where

$$\beta_1 = \frac{1}{2^{1/2}}\left[1 + (1 + \beta^2)^{1/2}\right]^{1/2}$$
$$\beta_2 = -\frac{\beta}{2^{1/2}[1 + (1 + \beta^2)^{1/2}]^{1/2}} \tag{22}$$

Recognizing Eq (19) as the modified Bessel equation of order one, the general solution for the complex vector potential A_2 is

$$A_2(r,z,t) = \{C_2 I_1(\alpha r) + C_3 K_1(\alpha r)\} \exp i(kz - \omega t) \tag{23}$$

where I_1 is the modified Bessel function of the first kind of order one, K_1 is the modified Bessel function of the second kind of order one, and C_2 and C_3 are arbitrary complex constants to be determined Similarly, the general solution of Eq (11) in the form $A_1(r) \exp i(kz - \omega t)$, and which satisfies condition (18), is

$$A_1(r,z,t) = C_1 K_1(kr) \exp i(kz - \omega t) \tag{24}$$

where C_1 is an arbitrary complex constant to be determined Finally, the general solution of Eq (12), satisfying condition (17), is

$$A_3(r,z,t) = C_4 I_1(\gamma r) \exp i(kz - \omega t) \tag{25}$$

where C_4 is an arbitrary complex constant to be determined and γ is the complex separation constant given by

$$\gamma = k\{1 - i\mu_3\sigma_3\omega/k^2\}^{1/2} = k(1 - i\epsilon)^{1/2} \tag{26}$$

In Eq (26), the quantity $\epsilon = \mu_3\sigma_3\omega/k^2$ represents the core magnetic Reynolds number based on wave speed and wavelength Defining the complex separation constant γ by $\gamma = \gamma_1 + i\gamma_2$, then from (26) we obtain for the real and imaginary parts

$$\gamma_1 = k\epsilon_1 \qquad \gamma_2 = k\epsilon_2 \tag{27}$$

where

$$\epsilon_1 = \frac{1}{2^{1/2}}\left[1 + (1 + \epsilon^2)^{1/2}\right]^{1/2}$$
$$\epsilon_2 = -\frac{\epsilon}{2^{1/2}[1 + (1 + \epsilon^2)^{1/2}]^{1/2}} \tag{28}$$

Substituting the solutions (23–25) into the boundary conditions (13–16), yields the following four linear algebraic equations for the four complex constants, C_1, C_2, C_3, and C_4:

$$C_1 K_1(kR_0) - C_2 I_1(\alpha R_0) - C_3 K_1(\alpha R_0) = 0 \tag{29}$$

$$C_1[kK_0(kR_0)] - C_2[\alpha I_0(\alpha R_0)] - C_3[\alpha K_0(\alpha R_0)] = -\mu_0 NI \tag{30}$$

$$C_2 I_1(\alpha R_1) + C_3 K_1(\alpha R_1) - C_4 I_1(\gamma R_1) = 0 \tag{31}$$

$$C_2[\alpha I_0(\alpha R_1)] + C_3[\alpha K_0(\alpha R_1)] - C_4\delta[\gamma I_0(\gamma R_1)] = 0 \tag{32}$$

where the identities $(1/\xi)(d/d\xi)(\xi I_1) = I_0$ and $(1/\xi)(d/d\xi)(\xi K_1) = K_0$ have been used

Solving for the constants C_2 and C_3, we obtain

$$C_2 = \frac{(\mu_0 NI)M_1}{M_2M_3 + M_4M_5} \qquad C_3 = \frac{(\mu_0 NI)M_6}{M_2M_3 + M_4M_5} \tag{33}$$

where

$$M_1 = K_1(kR_0)[\delta\gamma K_1(\alpha R_1)I_0(\gamma R_1) - \alpha I_1(\gamma R_1)K_0(\alpha R_1)]$$
$$M_2 = \delta\gamma K_1(\alpha R_1)I_0(\gamma R_1) - \alpha I_1(\gamma R_1)K_0(\alpha R_1)$$
$$M_3 = \alpha I_0(\alpha R_0)K_1(kR_0) - kI_1(\alpha R_0)K_0(kR_0)$$
$$M_4 = \alpha I_1(\gamma R_1)I_0(\alpha R_1) - \delta\gamma I_1(\alpha R_1)I_0(\gamma R_1)$$
$$M_5 = \alpha K_0(\alpha R_0)K_1(kR_0) - kK_1(\alpha R_0)K_0(kR_0)$$
$$M_6 = K_1(kR_0)[\alpha I_1(\gamma R_1)I_0(\alpha R_1) - \delta\gamma I_1(\alpha R_1)I_0(\gamma R_1)]$$

Thus, the exact solution for the complex vector potential in the annular region 2 is given by Eq (23) with C_2 and C_3 given by Eq (33) Using Eq (23), the magnetic field components, the electric field, and the current density are determined from Eqs (6, 8, and 9), respectively Since we shall be concerned principally with an examination of the induced axial body force which involves the product of the radial component of the magnetic field and the current density, we have, in particular,

$$B_{2r} = -\frac{\partial A_2}{\partial z} = -ikA_2$$
$$j_2 = -\sigma_2\left(\frac{\partial A_2}{\partial t} + U\frac{\partial A_2}{\partial z}\right) = i\sigma_2\omega S A_2 \tag{34}$$

where B_2 and j_2 are complex quantities

B Asymptotic Solution for the Case When $kR_1 \ll 1$, $|\alpha R_1| \ll 1$, and $|\gamma R_1| \ll 1$ Simultaneously

We use the following asymptotic forms[11] for the four different Bessel functions involved in the foregoing solution for A_2 when the absolute value of the argument is very much less than one:

$$I_0(z) \sim 1 \qquad I_1(z) \sim z/2$$
$$K_0(z) \sim -\{\ln(z/2) + \rho\}$$
$$K_1(z) \sim -z/2\{\ln(z/2) + \rho - \tfrac{1}{2}\} - (1/z)$$

where ρ = Euler's constant Substituting into Eq (33), C_2 and C_3 reduce to

$$C_2 \sim \frac{\mu_0 NI}{2} \qquad C_3 \sim -\frac{(\mu_0 NI)R_1^2\alpha(1-\delta)}{2\delta}$$

The complex vector potential of Eq (23) reduces to

$$A_2(r,z,t) \sim \frac{\mu_0 NI}{2}\left[r + \frac{R_1^2(1-\delta)}{\delta r}\right]\exp i(kz - \omega t) \tag{35}$$

C Asymptotic Solution for the Case When $kR_1 \gg 1$, $|\alpha R_1| \gg 1$, and $|\gamma R_1| \gg 1$ Simultaneously

When the absolute values of the argument of the Bessel functions are very much greater than one, the functions reduce to the following asymptotic forms[11]:

$$I_0(z) \sim \frac{e^z}{(2\pi z)^{1/2}} \qquad I_1(z) \sim \frac{e^z}{(2\pi z)^{1/2}}$$
$$K_0(z) \sim \left(\frac{\pi}{2z}\right)^{1/2} e^{-} \qquad K_1(z) \sim -\left(\frac{\pi}{2z}\right)^{1/2} e^{-z}$$

Substituting the foregoing asymptotic forms into Eq (33), C_2 and C_3 reduce to

$$C_2 = \frac{(\mu_0 NI)(2\pi\alpha R_0)^{1/2}e^{-\alpha R_0}(\alpha + \delta\gamma)}{(\alpha + k)(\alpha + \delta\gamma) - e^{-2\alpha(R_0 - R_1)}(\alpha - k)(\alpha - \delta\gamma)} \tag{36}$$

$$C_3 = -\frac{(\mu_0 NI)(2\alpha R_0/\pi)^{1/2}e^{\alpha R_0}(\alpha - \delta\gamma)}{e^{2\alpha(R_0 - R_1)}(\alpha + k)(\alpha + \delta\gamma) - (\alpha - k)(\alpha - \delta\gamma)}$$

Using the foregoing asymptotic forms for $I_1(\alpha r)$ and $K_1(\alpha r)$ and the forementioned expressions for C_2 and C_3, Eq (23) reduces to

$$A_2(r,z,t) \sim (\mu_0 NI)\left(\frac{R_0}{r}\right)^{1/2}\exp i(kz - \omega t)\left\{\frac{(\alpha + \delta\gamma)\exp[-\alpha(R_0 - r)] + (\alpha - \delta\gamma)\exp[-\alpha(R_0 - 2R_1 + r)]}{(\alpha + k)(\alpha + \delta\gamma) - (\alpha - k)(\alpha - \delta\gamma)\exp[-2\alpha(R_0 - R_1)]}\right\} \tag{37}$$

Each electromagnetic variable sought is represented by the real part of its corresponding complex quantity Now, to use the complex vector potential A_2 to derive expressions for the real parts, it is necessary that the quantity in the braces of Eq (37) be explicitly expressed in the form $W_1 + iW_2$ where W_1 and W_2 are real Remembering that $\alpha = k\beta_1 + ik\beta_2$ and $\gamma = k\epsilon_1 + ik\epsilon_2$, and rationalizing in the usual way by multiplying numerator and denominator by the complex conjugate of the denominator, we obtain, after some lengthy calculation,

$$A_2(r,z,t) \sim (\mu_0 NI)\left(\frac{R_0}{r}\right)^{1/2}\exp i(kz - \omega t) \times \left\{\frac{(P_1P_3 + P_2P_4) + i(P_1P_4 - P_2P_3)}{P_1^2 + P_2^2}\right\} \tag{38}$$

where

$$P_1 = k\{[1 + \beta_1][kR_1(\beta_1 + \delta\epsilon_1)] - \beta_2[kR_1(\beta_2 + \delta\epsilon_2)]\} + k\exp\left[-2kR_1\beta_1\left(\frac{R_0}{R_1} - 1\right)\right]\{[\beta_1 - 1][-kR_1(\beta_1 - \delta\epsilon_1)] - \beta_2[kR_1(\delta\epsilon_2 - \beta_2)]\}\cos\left[2kR_1\beta_2\left(\frac{R_0}{R_1} - 1\right)\right] + k\exp\left[-2kR_1\beta_1\left(\frac{R_0}{R_1} - 1\right)\right]\{\beta_2[-kR_1(\beta_1 - \delta\epsilon_1)] + [\beta_1 - 1][kR_1(\delta\epsilon_2 - \beta_2)]\}\sin\left[2kR_1\beta_2\left(\frac{R_0}{R_1} - 1\right)\right]$$

$$P_2 = k\{\beta_2[kR_1(\beta_1 + \delta\epsilon_1)] + [\beta_1 + 1][kR_1(\beta_2 + \delta\epsilon_2)]\} + k\exp\left[-2kR_1\beta_1\left(\frac{R_0}{R_1} - 1\right)\right]\{\beta_2[-kR_1(\beta_1 - \delta\epsilon_1)] + [\beta_1 - 1][kR_1(\delta\epsilon_2 - \beta_2)]\}\cos\left[2kR_1\beta_2\left(\frac{R_0}{R_1} - 1\right)\right] - k\exp\left[-2kR_1\beta_1\left(\frac{R_0}{R_1} - 1\right)\right]\{[\beta_1 - 1][-kR_1(\beta_1 - \delta\epsilon_1)] - \beta_2[kR_1(\delta\epsilon_2 - \beta_2)]\}\sin\left[2kR_1\beta_2\left(\frac{R_0}{R_1} - 1\right)\right]$$

$$P_3(r) = \exp\left[-kR_1\beta_1\left(\frac{R_0}{R_1} - \frac{r}{R_1}\right)\right]\left\{[kR_1(\beta_1 + \delta\epsilon_1)] \times \cos\left[kR_1\beta_2\left(\frac{R_0}{R_1} - \frac{r}{R_1}\right)\right]\right\} + \exp\left[-kR_1\beta_1\left(\frac{R_0}{R_1} - \frac{r}{R_1}\right)\right] \times \left\{[kR_1(\beta_2 + \delta\epsilon_2)]\sin\left[kR_1\beta_2\left(\frac{R_0}{R_1} - \frac{r}{R_1}\right)\right]\right\} - \exp\left[-kR_1\beta_1\left(\frac{R_0}{R_1} - 2 + \frac{r}{R_1}\right)\right] \times \left\{[-kR_1(\beta_1 - \delta\epsilon_1)]\cos\left[kR_1\beta_2\left(\frac{R_0}{R_1} - 2 + \frac{r}{R_1}\right)\right]\right\} - \exp\left[-kR_1\beta_1\left(\frac{R_0}{R_1} - 2 + \frac{r}{R_1}\right)\right]\left\{[kR_1(\delta\epsilon_2 - \beta_2)] \times \sin\left[kR_1\beta_2\left(\frac{R_0}{R_1} - 2 + \frac{r}{R_1}\right)\right]\right\}$$

$$P_4(r) = \exp\left[-kR_1\beta_1\left(\frac{R_0}{R_1} - \frac{r}{R_1}\right)\right]\left\{[kR_1(\beta_2 + \delta\epsilon_2)] \times \cos\left[kR_1\beta_2\left(\frac{R_0}{R_1} - \frac{r}{R_1}\right)\right]\right\} - \exp\left[-kR_1\beta_1\left(\frac{R_0}{R_1} - \frac{r}{R_1}\right)\right] \times \left\{[kR_1(\beta_1 + \delta\epsilon_1)]\sin\left[kR_1\beta_2\left(\frac{R_0}{R_1} - \frac{r}{R_1}\right)\right]\right\} - \exp\left[-kR_1\beta_1\left(\frac{R_0}{R_1} - 2 + \frac{r}{R_1}\right)\right]\left\{[kR_1(\delta\epsilon_2 - \beta_2)] \times\right.$$

$$\cos\left[kR_1\beta_2\left(\frac{R_0}{R_1} - 2 + \frac{r}{R_1}\right)\right]\Bigg\} +$$

$$\exp\left[-kR_1\beta_1\left(\frac{R_0}{R_1} - 2 + \frac{r}{R_1}\right)\right]\Bigg\{ [-kR_1(\beta_1 - \delta\epsilon_1)] \times$$

$$\sin\left[kR_1\beta_2\left(\frac{R_0}{R_1} - 2 + \frac{r}{R_1}\right)\right]\Bigg\}$$

where P_3 and P_4 depend on r, whereas P_1 and P_2 do not

IV Induced Axial Body Force

The Lorentz force per unit volume acting on a current-carrying element is $\mathbf{F} = \mathbf{j} \times \mathbf{B}$ For region 2, in terms of real quantities, $\mathbf{F}_2 = (R_e\mathbf{j}_2) \times (R_e\mathbf{B}_2)$ Carrying out the cross product, we have, in component form, $F_{2z} = -(R_ej_2)(R B_2)$ and $F_{2r} = (R_ej_2)(R B_{2z})$ In particular, using Eq (34),

$$F_2 = \sigma_2\omega kS[R(iA_2)]^2 \tag{39}$$

Using Eq (39), we now develop expressions for the induced axial body force for the two limiting cases just considered

Case a: $kR_1 \ll 1$, $|\alpha R_1| \ll 1$ and $|\gamma R_1| \ll 1$ simultaneously Substituting Eq (35) into Eq (39), we obtain

$$F_2 \sim \frac{(\mu_0 NI)^2}{4}(\sigma_2\omega kS)\left[r + \frac{R_1^2(1-\delta)}{r\delta}\right]^2 \times \cos^2\left(kz - \omega t + \frac{\pi}{2}\right) \tag{40}$$

where $R_1 \leq r \leq R_0$

The axial force averaged over a cycle is

$$\langle F_2(r)\rangle \sim \frac{\mu_0}{8}(NI)^2k^3\beta\left[r + \frac{R_1^2(1-\delta)}{r\delta}\right]^2 \tag{41}$$

where we have used $(\mu_0\sigma_2\omega/k^2)S = \beta$ In particular, at the inner wall $r = R_1$, Eq (41) becomes

$$\langle F_{2z}(R_1)\rangle \sim \frac{\mu_0}{8R_1}(NI)^2(kR_1)^3\beta\frac{1}{\delta^2} \tag{42}$$

It is seen from Eq (41) that the induced axial body force at any station in the annulus, for the limiting case $kR_1 \ll 1$, $|\alpha R_1| \ll 1$, and $|\gamma R_1| \ll 1$ simultaneously, is directly proportional to β Thus, when $S < 0$ then $\beta < 0$, and therefore $\langle F_2\rangle < 0$ corresponding to generator action When β or S is positive, acceleration occurs It is further noted that the axial force is proportional to the cube of the wave number, independent of the core magnetic Reynolds number ϵ and, at the inner wall, inversely proportional to the square of the permeability ratio

From the practical viewpoint, the small $kR_1 \ll 1$ etc solution is quite uninteresting Given the current sheet input NI and geometry R_1, the force at R_1, for example, varies as the cube of (kR_1) [see Eq (42)], so that operating at large wavelengths produces small forces Let us, therefore, proceed to examine in detail the second and more significant of the asymptotic forms

Case b: $kR_1 \gg 1$, $|\alpha R_1| \gg 1$ and $|\gamma R_1| \gg 1$ simultaneously Substituting Eq (38) into Eq (39), we obtain

$$F_2 \sim (\mu_0 NI)^2\left(\frac{R_0}{r}\right)\sigma_2\omega kS\{L_1 + L_2 + L_3\} \tag{43}$$

where

$$L_1 = \frac{(P_1P_3 + P_2P_4)^2\cos^2(kz - \omega t + \pi/2)}{[P_1^2 + P_2^2]^2}$$

$$L_2 = -\frac{2(P_1P_3 + P_2P_4)(P_1P_4 - P_2P_3)\sin(kz - \omega t + \pi/2)\cos(kz - \omega t + \pi/2)}{[P_1^2 + P_2^2]^2}$$

$$L_3 = \frac{(P_1P_4 - P_2P_3)^2\cos^2(kz - \omega t + \pi/2)}{[P_1^2 + P_2^2]^2}$$

The body force averaged over a cycle is

$$\langle F_2(r)\rangle \sim \frac{(\mu_0NI)^2}{2}\left(\frac{R_0}{r}\right)(\sigma_2\omega kS)\left\{\frac{P_3^2(r) + P_4^2(r)}{P_1^2 + P_2^2}\right\} \tag{44}$$

Examining the expressions for P_1 and P_2, it is noted that k is a common multiplier of each term in the expressions both for P_1 and P_2 Hence, defining $P_1 = kT_1$ and $P_2 = kT_2$ and recalling that $(\mu_0\sigma_2\omega/k^2)S = \beta$, Eq (44) becomes

$$\langle F_2(r)\rangle \sim \frac{\mu_0}{2r}(NI)^2\beta(kR_1)\left(\frac{R_0}{R_1}\right)\left[\frac{P_3^2(r) + P_4^2(r)}{T_1^2 + T_2^2}\right] \tag{45}$$

Evaluating Eq (45) at the inner wall $r = R_1$ and remembering that β_1 and β_2 depend on β [i e , see Eq (22)], and ϵ_1 and ϵ_2 depend on ϵ [see Eq (28)], Eq (45) becomes

$$\langle F_{2z}(R_1)\rangle \sim \frac{\mu_0}{2R_1}(NI)^2G\left\{\frac{R_0}{R_1},(kR_1),\delta,\beta,\epsilon\right\} \tag{46}$$

where

$$G\left\{\frac{R_0}{R_1},(kR_1),\delta,\beta,\epsilon\right\} = \beta(kR_1)\left(\frac{R_0}{R_1}\right)\left[\frac{P_3^2(R_1) + P_4^2(R_1)}{T_1^2 + T_2^2}\right] \tag{47}$$

With the current sheet input NI and inner radius R_1 preassigned, it is seen from Eq (47) that the induced axial body force at the inner wall is a function only of the five nondimensional parameters $R_0/R_1,(kR_1),\delta,\beta,$ and ϵ In the next section, we shall examine the variation of G with each of the five parameters From the very nature of the skin effect, it should be emphasized here that the behavior of the induced axial body force at R_1 as a function of these five parameters reflects exactly the behavior at every radial position in the annular region and even that of the total integrated force across the annulus Thus, for example, the particular combination of parameters which maximizes the force at R_1 will maximize the force at every other radial position, and hence, the total integrated force over the cross section Finally, since the quantity in the square brackets of Eq (45) is positive, it is again observed that the force is retarding when β (or S) is negative and accelerating when β (or S) is positive

V Numerical Results and Discussion

The behavior of the dimensionless axial body force, evaluated at $r = R_1$ [i e , G of Eq (47)], with respect to the five dimensionless parameters $R_0/R_1,(kR_1),\delta,\beta,$ and ϵ, was calculated using the Avco IBM 7094 computer The results are shown in Figs 2–8

Let us first consider the behavior of G as a function of β (i e , the product of the fluid magnetic Reynolds number based on wave speed and wavelength multiplied by the slip), for various values of the other parameters Figures 2–4 show the variation of G with β for the three radius ratio cases $R_0/R_1 = 1\ 5$, $1\ 2$, and $1\ 1$, respectively, for the value $(kR_1) = 10$ Four curves are displayed in each figure, each curve representing a (δ,ϵ) combination of particular significance Thus, the curve labeled $\delta = 1$, $\epsilon = 0$, corresponds to operation in an annulus with a hollow core The curve labeled $\delta = 1$, $\epsilon = \beta$, corresponds to fluid injection and acceleration over the entire cross section The curve labeled $\delta = 10^{-3}$, $\epsilon = 10^6$ corresponds to operating in an annulus with a core whose permeability is a thousand times that of free space but in which eddy currents are free to flow, and finally the case $\delta =$

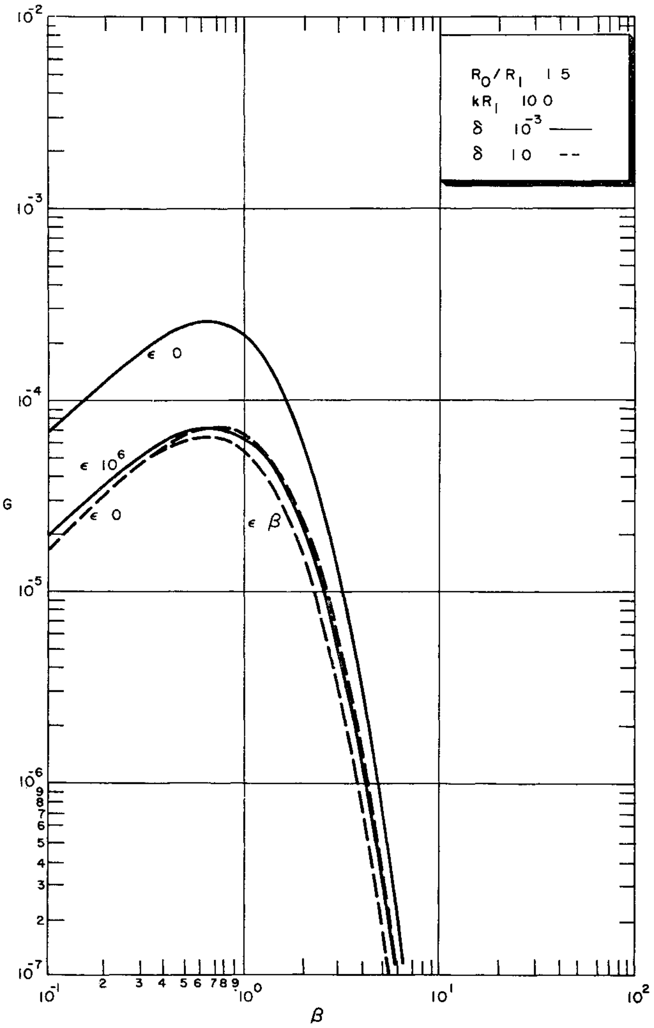

Fig 2 Asymptotic behavioı of dimensionless axial body force evaluated at the inneı radius with fluid magnetic Reynolds number (based on wave speed and wavelength) multiplied by the slip

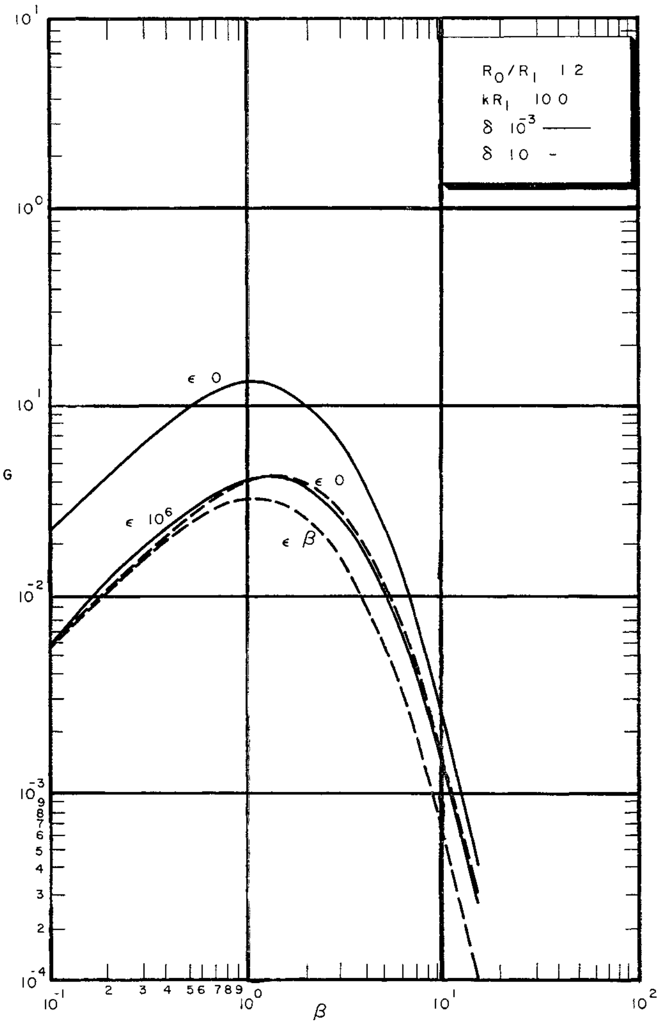

Fig 3 Asymptotic behavior of dimensionless axial body force evaluated at the inner radius with fluid magnetic Reynolds numbeı (based on wave speed and wavelength) multiplied by the slip

10^{-3}, $\epsilon = 0$ corresponds to the preceding case, but with the core material sufficiently laminated so that no circulating eddy currents are permitted to flow From Figs 2–4, the following six observations can be made:

1) The over-all similar behavior with β of the four different (δ,ϵ) curves in each figure may be noted That is, for each curve, G varies linearly with β for small β, reaches a peak, and then falls exponentially with increasing β The position of the corresponding maximum moves to the right with decreasing R_0/R_I

2) By comparing the behavior at low β (i e , the linear increase of G with β), with the behavior across the remainder of the β spectrum, one notes how vividly displayed is the phenomenon of the skin effect The low β behavior is the one usually assumed when naively neglecting the skin effect Hence, from the point of view of application, one would argue that all that is required to increase the electromechanical coupling is to increase the value of β indefinitely However, beyond the peak G, the conducting fluid rapidly begins excluding more and more of the electromagnetic field so that the induced forces begin to fall exponentially with large β

3) Consistent with the first conjecture made in the introduction, the existence of a peak force for a certain value of β does suggest that the best one could hope for (having once decided on the nature of the core) is to adjust the transmission line parameters (k,ω) and those of the conducting fluid (σ,U) so that operation occurs at the value of β which maximizes G

4) Consistent with the third conjecture made in the introduction, one notes the significant increase in G, at peak G, when operating with a laminated core of high permeability (e g , the case $\delta = 10^{-3}$, $\epsilon = 0$) At peak G, G increases roughly by a factor of four for the case $R_0/R_I = 1$ 5, the factor decreasing slowly with decreasing radius ratio

5) Comparing the magnitude of G, at corresponding β values, with increasing R_0/R_I illustrates the absurdity of operating over the entire cross section as compared with annular operation For example, it is noted, comparing the $R_0/R_I = 1$ 5 curves with those for $R_0/R_I = 1$ 1, that even for very low β's (where the skin effect is weak), the corresponding forces have decreased by more than three orders of magnitude, and for β's approximately five or so, the forces have decreased by six to seven orders of magnitude Thus, it is only the small annulus of fluid near the coils that actually experience any significant inductive effects, and, consistent with the second conjecture of the introduction, no advantage is gained in attempting to accelerate (or decelerate) more fluid by injecting over the entire cross section

6) Excluding the exceptional case $\delta = 10^{-3}$, $\epsilon = 0$, the slight differences in the magnitude of G among the remaining three curves in each figure is attributable to the counter balancing roles played by eddy currents and permeability If eddy currents are permitted to flow, the mutual induction of all the circulating currents in the region $r < R_I$ on any ring circuit $R_0 \geq r \geq R_I$ would have the effect of decreasing the total flux through the ring r so that the magnitude of the induced current there would be less than it would be if no core

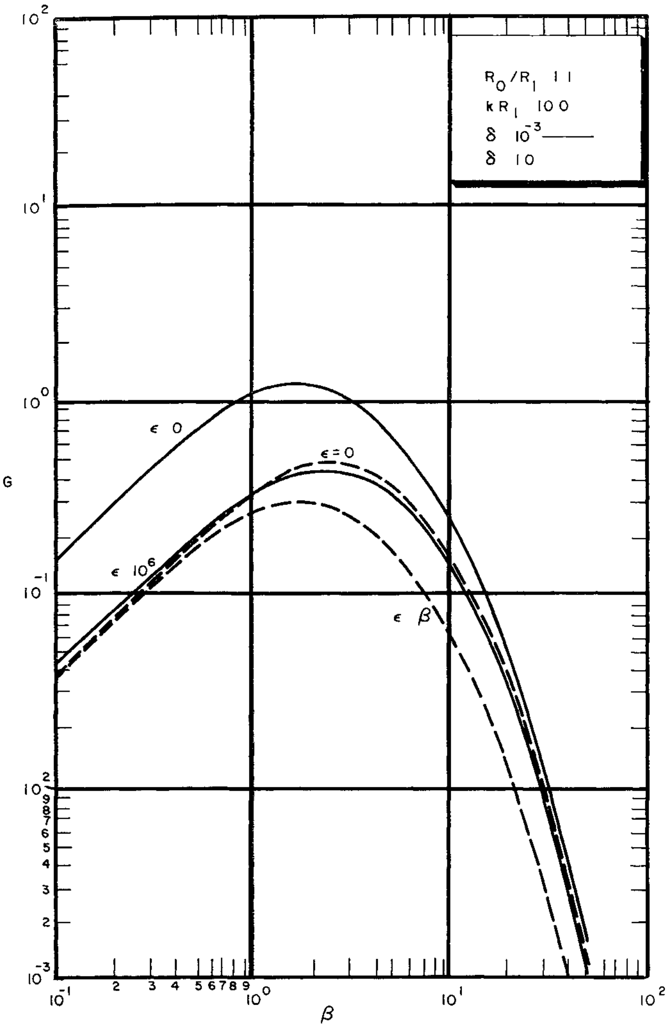

Fig 4 Asymptotic behavior of dimensionless axial body force evaluated at the inner radius with fluid magnetic Reynolds number (based on wave speed and wavelength) multiplied by the slip

currents were permitted to flow (or if the core were hollow) On the other hand, since large permeability increases the flux of the radial component of the magnetic field, the force ultimately induced would depend on which of these two effects succeeded in outbalancing the other For example, consider Fig 4 at $\beta = 0\ 5$ The value of G for the case $\delta = 1\ 0$, $\epsilon = \beta$ is less than that for the case $\delta = 1$, $\epsilon = 0$ This is so because the mutual effects of the small currents still circulating in the fluid region $r < R_1$, decreases the net flux through the ring $r = R_1$ so that the current induced at $r = R_1$ is less than it would be were the core hollow (i e , $\delta = 1$, $\epsilon = 0$) On the other hand, the insertion of the core $\delta = 10^{-3}$, $\epsilon = 10^6$ has the effect of increasing the value of G only slightly above the case $\delta = 1\ 0$, $\epsilon = 0$ In this case, the increased B_2 due to the higher permeability slightly overpowers the stronger mutual induction effects of the larger circulating core currents so that the force induced turns out to be slightly larger Further, the existence of a crossover point is to be noted (i e , at $\beta \sim 1$) where this latter situation reverses itself

Figures 5–8 are mainly elaborations on the six observations just discussed Figure 5 represents the variation of G with δ for the particular values of R_0/R_1 just discussed keeping the remaining three parameters fixed For each R_0/R_1, G is proportional to some negative power of δ for large δ and then asymptotically approaches a finite limit as the permeability of the core material goes to infinity Figure 6 shows the variation of G with R_0/R_1 for the various (δ, ϵ) combinations just used keeping the remaining parameters fixed The general

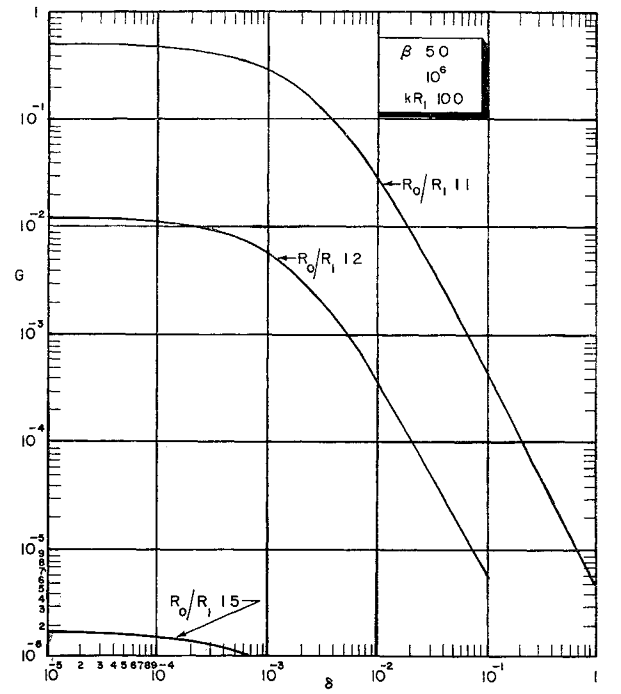

Fig 5 Asymptotic behavior of dimensionless axial body force evaluated at the inner radius with ratio of free space to core permeability

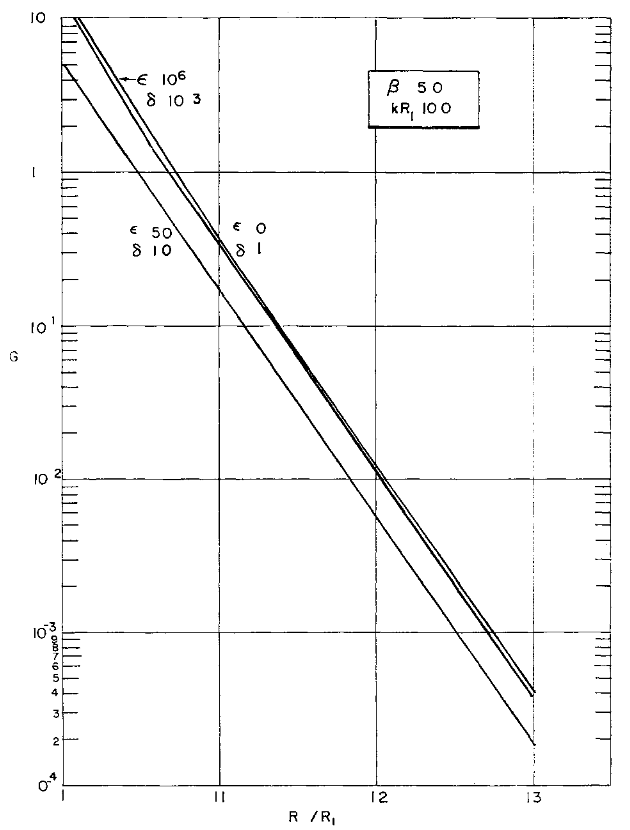

Fig 6 Asymptotic behavior of dimensionless axial body force evaluated at the inner radius with annulus radius ratio

behavior is marked by a very rapid exponential decrease of G with distance from the coil To be noted once again is the marked role played by the skin effect A decrease in depth of approximately 10% from R_0/R_1 of 1 1 to 1 2, attenuates the force by a factor of the order of thirty

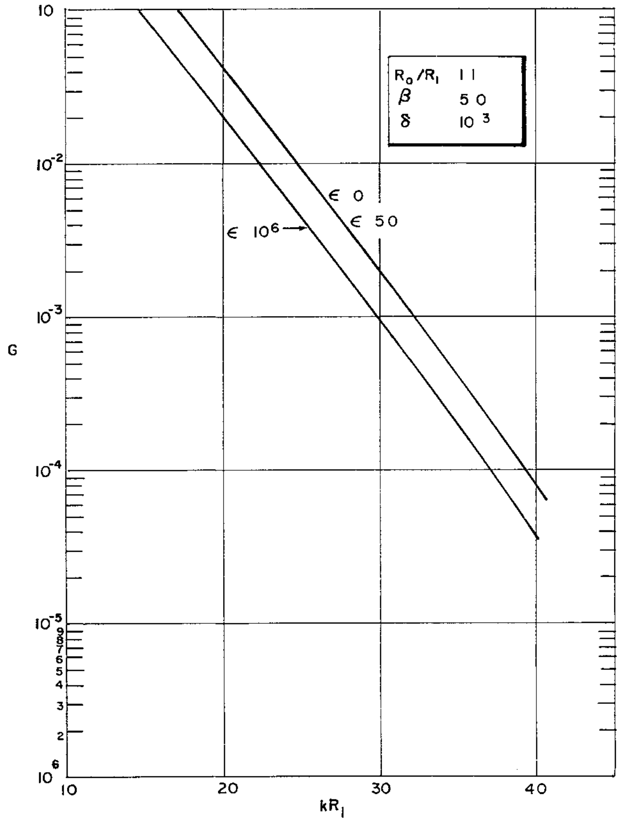

Fig 7 Asymptotic behavior of dimensionless axial body force evaluated at the inner raduis with product of wave-number and inner radius

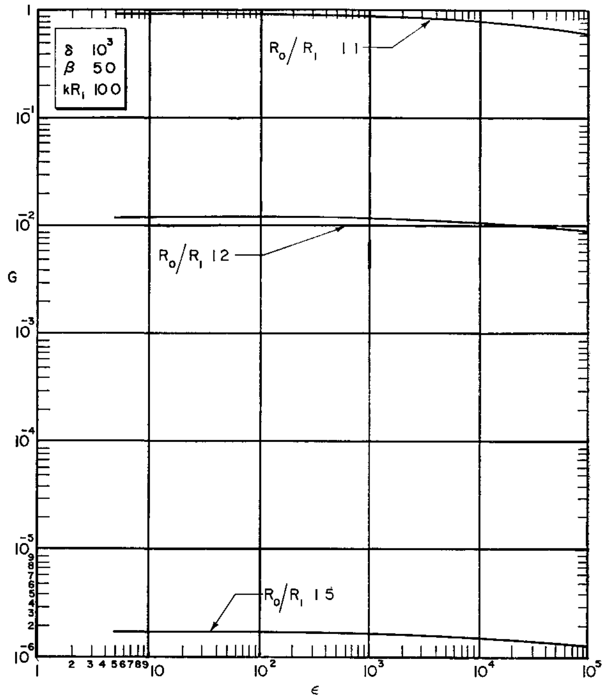

Fig 8 Asymptotic behavior of dimensionless axial body force evaluated at the inner radius with core magnetic Reynolds number (based on wave speed and wavelength)

Figure 7 is a plot of G against (kR_1) for various values of ϵ keeping the remaining three parameters fixed As the ratio of the inner radius to the operating wave length increases, G rapidly decreases exponentially Operating a traveling wave device at the smallest value of k possible (i e , consistent with the condition $kR_1 \gg 1$) to produce large G's may or may not prove advantageous This is so because G also depends on β which in turn is inversely proportional to k^2, and we have already observed the attenuating effect on G when β is made too large However, the foregoing remarks do suggest the existence of definite values for (kR_1) and β which render G a maximum [As far as the remaining parameters are concerned, R_0/R_1 is preassigned while one chooses and laminates ($\epsilon \sim 0$) the best available magnetic material ($\delta \sim 0$)] The optimizing values of the parameters (kR_1) and β, and the corresponding maximum value of G could, in principal, be obtained analytically by applying the usual methods of the differential calculus to the function G Figure 8 represents the variation of G with ϵ for various values of R_0/R_1 keeping the remaining three parameters fixed Note that G decreases only slightly with ϵ over the range of ϵ spanning five orders of magnitude

In conclusion, it should be remarked that the results for the intermediate case, i e , kR_1, $|\alpha R_1|$, and $|\gamma R_1|$ of order one simultaneously, require making full use of the Bessel functions of complex argument, and are not discussed here

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